Complex group velocity and energy transport in absorbing media

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Complex group velocity is common in absorbing and active media, yet its precise physical meaning is unclear. While in the case of a nondissipative medium the group velocity of propagating waves $C_g=d\omega/dk$ is exactly equal to the observable energy velocity (defined as the ratio between the energy flux and the total energy density) $C_g=\bar{F}/\bar{E}$, in a dissipative medium $C_g=d\omega/dk$ is in general a complex quantity which cannot be associated with the velocity of energy transport. Nevertheless, we find that the complex group velocity may contain information about the energy transport as well as the energy dissipated in the medium. The presented analysis is intended to expound the connection between the complex group velocity and energy transport characteristics for a class of hyperbolic dissipative dynamical systems. Dissipation mechanisms considered herein include viscous and viscoelastic types of damping. Both cases of spatial and temporal decay are discussed. The presented approach stems from the Lagrangian formulation and is illustrated with identities that relate the complex group velocity and energy transport characteristics for the damped Klein-Gordon equation; Maxwell's equations, governing electromagnetic waves in partially conducting media; and Biot's theory, governing acoustic wave propagation in porous solids.

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I. INTRODUCTION

The equivalence of the group velocity and the velocity of energy transport is known to be a classical result in the case of conservative dispersive media [1–4]. In this case, the group velocity C_g equals the energy velocity C_e, and thus, can be defined both kinematically, $C_g=\partial\omega/\partial k$, and from energy principles, as the ratio between the average energy flux $\bar{F}$ and the average mechanical energy $\bar{E}$. According to Whitham's average variational principle [3] and Lighthill's theorem [4], the relation

$$C_g = C_e = \frac{\bar{F}}{\bar{E}}, \tag{1}$$

is valid in the very general case of dispersive uniform lossless media. In the case of nonuniform and anisotropic media, with some restrictions of a geometric nature, the principle of eigenvalue perturbations, suggested by Biot [2], proves the identity. The latter heuristic approach does not prove (1) in general, but rather establishes the validity of the procedure in a wide variety of cases.

In the case of absorbing media, the energy velocity is defined similarly to conservative systems as the ratio between the average energy flux and average total mechanical energy [1,5–8], however, the kinematically defined group velocity is in general complex and obviously can no longer be associated with the velocity of energy transport,

$$C_g \neq C_e = \frac{\bar{F}}{\bar{E}}. \tag{2}$$

Unlike conservative systems, the interpretation of the complex group velocity from energy principles is not established. The physical meaning of the complex group velocity thus remains obscure, so that it is sometimes referred as a quantity with only an abstract mathematical meaning [8,9].

On the other hand, from a kinematic point of view, the real and imaginary parts of the complex group velocity $C_g=\partial\omega/\partial k$ (with a real value of a wave number) play an important role in the description of the behavior of a spatially localized Gaussian wave packet traveling in absorbing media [9,10]. According to the saddle point approximation, the central wave number k_c is in general not conserved, but experiences a drift which is directly proportional to the imaginary part of the group velocity [10],

$$x_M = \mathrm{Re}\left(\left.\frac{d\omega}{dk}\right|_{k_M} \right) t, \tag{3}$$

$$k_M = k_c + \Delta^2\ \mathrm{Im}\left(\left.\frac{d\omega}{dk}\right|_{k_M} \right) t, \tag{4}$$

where Δ corresponds to the width of the Gaussian. If a medium is dissipation free, then $\mathrm{Im}(d\omega/dk)=0$, and the spatial maximum of the wave packet moves with the group velocity evaluated at the central wave number k_c, $x_M=C_g(k_c)t$ [3,10]. As will be discussed in the following, examples of dissipative media with purely real group velocities are possible, in which case the central wave number k_c is conserved, and the location of the spatial maximum x_M can be found similarly to conservative systems.

The present analysis is intended to investigate the possibility of extrapolation of the energetic definition (1) of the complex group velocity to the case of absorbing media, and examples of dynamical systems whose group velocities allow energetic interpretations are presented.

The general analysis is limited to coupled hyperbolic nonconservative Lagrangian dynamical systems with viscous type dissipation. The Lagrangian densities are described by corresponding quadratic forms and the dissipation is included by the corresponding Rayleigh pseudopotential. The

resulting governing equations represent a system of coupled linear Klein-Gordon equations with damping (KGD) and yield the conservation form

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + 2D = 0, \tag{5}$$

where D denotes viscous dissipation function.

In general, equipartition of energy does not hold in absorbing media. The modewise energy balance equations are derived to establish the connection between the kinetic energy, potential energy and energy losses. These relations replace the equipartition of energy principle and provide the foundation for further analysis.

The development, which to some extent follows Biot's approach for conservative systems [2], allows one to interpret complex group velocities of spatially damped waves in terms of the following average quantities: energy flux $\bar{F}$, kinetic energy $\bar{T}$, and energy losses $\bar{\mathcal{E}}$. In the case of temporally damped waves, group velocities are directly related to the velocities of energy transport. For example, in the case of single one-dimensional KGD equation group velocities of the spatially and temporally damped waves admit the following energetic interpretations, respectively,

$$C_g = \frac{\bar{F} + ic_\phi \bar{\mathcal{E}}}{2\bar{T} + i\bar{\mathcal{E}}}, \quad C_g = \frac{|\omega|^2}{\omega_R^2} C_e. \tag{6}$$

where $c_\phi = \omega / k_R$ denotes the phase velocity. It is important to stress that in the limiting case of no dissipation both resulting expressions (6) recover the energetic definition (1) for conservative systems. In the very general case of coupled KGD equations energetic representations (6) only hold approximately (modewise). The validity of approximations is subsequently discussed and illustrated.

Section II provides the necessary preliminaries and establishes the relations necessary for further development. In Sec. III modewise energy balance equations are derived to characterize the partition between the kinetic energy, potential energy, and energy losses. Section IV is dedicated to the interpretation of the complex group velocity from energy principles. The results are presented for both spatially and temporally damped waves. The following sections contain case studies. The identities are established for the three-dimensional version of the Klein-Gordon equation with damping in Sec. V, and for electromagnetic waves in a partially conducting medium in Sec. VI. The results for a multiphase poroelastic continuum in the framework of Biot's theory, including the high-frequency range theory, are given detailed consideration in Sec. VII. The conclusions are summarized in Sec. VIII.

II. WAVE MOTION AND LAGRANGIAN FORMALISM

Consider the one-dimensional Lagrange equation [11],

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial q_{k,x}} - \frac{\partial \mathcal{L}}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = 0, \tag{7}$$

where $\mathcal{L} = \mathcal{L}(q, \dot{q}_k, q_{k,x}, x, t)$ is the volumetric Lagrangian density, and D is the Rayleigh dissipation pseudopotential. Let the dynamical system with N degrees of freedom be governed by the following positive definite quadratic forms with corresponding symmetric matrices, so that kinetic and potential energies, T and V, dissipation pseudopotential D (possibly positive semidefinite), are given by,

$$2T = \rho_{ij}\dot{q}_i\dot{q}_j, \tag{8}$$

$$2V = \alpha_{ij}q_{i,x}q_{j,x} + \beta_{ij}q_iq_j, \tag{9}$$

$$2D = b_{ij}\dot{q}_i\dot{q}_j. \tag{10}$$

Here and henceforth the summation over repeated indices is implied, unless otherwise specified. With these definitions the Lagrange equation reads,

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial}{\partial x}\frac{\partial V}{\partial q_{k,x}} + \frac{\partial V}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = 0, \quad (k = 1 \ldots N), \tag{11}$$

and, for instance, in terms of continuum theories, provides a single governing equation for single phase media, $N=1$; two equations in the case of two-phase media, $N=2$; three equations for three-phase media, $N=3$ etc.

According to Eqs. (8)–(11) the governing equations represent the following hyperbolic system of partial differential equations,

$$\rho_{ij}\ddot{q}_j - \alpha_{ij}q_{j,xx} + \beta_{ij}q_j + b_{ij}\dot{q}_j = 0, \quad (i = 1 \ldots N), \tag{12}$$

with corresponding energy conservation form (5),

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2}\alpha_{ij}q_{i,x}q_{j,x} + \frac{1}{2}\beta_{ij}q_iq_j\right) + \frac{\partial}{\partial x}(-\alpha_{ij}\dot{q}_iq_{j,x}) + b_{ij}\dot{q}_i\dot{q}_j = 0, \tag{13}$$

which provides the expression for the energy flux F in terms of generalized coordinates,

$$F = -\alpha_{ij}\dot{q}_iq_{j,x}. \tag{14}$$

In the case of harmonic wave motion

$$q_j \sim A\exp[i(kx \pm \omega t)], \quad (j = 1 \ldots N),$$

the Lagrange Eq. (11) can be recast in the form analogous to [2],

$$-\omega^2\frac{\partial T'}{\partial q_k} + \frac{\partial V'}{\partial q_k} \pm i\omega\frac{\partial D'}{\partial q_k} = 0, \quad (k = 1 \ldots N), \tag{15}$$

where $2T' = \rho_{ij}q_iq_j$, $2V' = (\alpha_{ij}k^2 + \beta_{ij})q_iq_j$, $2D' = b_{ij}q_iq_j$, and provides the governing Eq. (12) in frequency wave-number space

$$[-\omega^2\rho_{ij} + a_{ij}(k) \pm i\omega b_{ij}]q_j = 0, \quad (i = 1 \ldots N), \tag{16}$$

where $a_{ij}(k) = \alpha_{ij}k^2 + \beta_{ij}$.

Dissipation leads to complex dispersion relations restricting further analysis either to the case of spatially damped waves so that real circular frequencies are mapped into the complex space of wave numbers $\omega \to k_R + ik_I$, or temporally damped waves, in which case real wave numbers are mapped into complex frequency space $k \to \omega_R + i\omega_I$. Thus, in the case of spatially (temporally) damped waves, for a given real ω

(real k) the system (16) represents one-parameter quadratic eigenvalue problem [12] for, in general, $2N$ complex eigenvalues k_n (eigenvalues ω_n) and up to $2N$ eigenvectors.

Multiplying the i^{th} Eq. (16) by $q_i^*/4$ one finds that in the case of spatially damped waves each of the eigenvalues k_n yields the following relation:

$$-\frac{1}{4}\rho_{ij}\omega^2 q_i^n q_j^{n*} + \frac{1}{4}a_{ij}(k_n)q_i^n q_j^{n*} \pm i\frac{1}{4}b_{ij}\omega q_i^n q_j^{n*} = 0, \quad (17)$$

where q_i^n represents i^{th} component of the eigenmode associated with the n^{th} eigenvalue k_n,

$$q_i^n = A_i^n(\omega)\exp[i(k_n x \pm \omega t)]. \quad (18)$$

In the case of the temporally damped waves all eigenvalues ω_n yield similar relations

$$-\frac{1}{4}\rho_{ij}\omega_n^2 q_i^n q_j^{n*} + \frac{1}{4}a_{ij}(k)q_i^n q_j^{n*} \pm i\frac{1}{4}b_{ij}\omega_n q_i^n q_j^{n*} = 0, \quad (19)$$

where q_i^n represents i^{th} component of the eigenmode associated with the n^{th} eigenvalue ω_n,

$$q_i^n = A_i^n(k)\exp[i(kx \pm \omega_n t)]. \quad (20)$$

Equations (17) and (19) constitute the foundation for further analysis.

III. MODEWISE ENERGY BALANCE AND ENERGY PARTITION

In the case of conservative systems ($b_{ij}=0$) both eigenvalues and eigenvectors are real, the above considerations lead to

$$-\frac{1}{4}\rho_{ij}\omega_n^2 q_i^n q_j^n + \frac{1}{4}a_{ij}(k)q_i^n q_j^n = 0, \quad (21)$$

and equipartition of energy follows immediately for each mode [2],

$$\bar{T}_n = \bar{V}_n. \quad (22)$$

For nonconservative dynamical systems under consideration equipartition of energy does not hold for spatially attenuated modes, however, it is found to be the case for temporally damped modes.

A. Spatially damped waves

Consider the case of spatially damped waves in detail. The values of the average over a period kinetic energy $\bar{T}_n$, potential energy $\bar{V}_n$, and viscous power dissipation $2\bar{D}_n$ associated with the n^{th} mode are provided in Appendix A. The expression for the modewise energy losses in one period is thus

$$2\bar{D}_n\frac{2\pi}{\omega} = \pi\omega b_{ij}q_i^n q_j^{n*}. \quad (23)$$

The following quantity, $\bar{\mathcal{E}}_n$, is introduced for further convenience to denote $1/4\pi$ fraction of viscous energy loss in one period,

$$\bar{\mathcal{E}}_n = \bar{D}_n/\omega = \frac{1}{4}\omega b_{ij}q_i^n q_j^{n*}. \quad (24)$$

According to the results of (A1) and (24) the expression (17) can be rewritten in the form of the modewise energy balance (no summation over repeated n's),

$$-\bar{T}_n + \Delta_n\bar{V}_n \pm i\bar{\mathcal{E}}_n = 0, \quad (25)$$

where the following complex, frequency dependent coefficient is introduced

$$\Delta_n = \frac{(\alpha_{ij}k_n^2 + \beta_{ij})q_i^n q_j^{n*}}{(\alpha_{ij}|k_n|^2 + \beta_{ij})q_i^n q_j^{n*}}. \quad (26)$$

The above relationship (25) replaces the energy equipartition principle for harmonic waves in conservative systems, $\bar{T}=\bar{V}$, which can be recovered from Eq. (25) in the limiting case of no dissipation ($\bar{\mathcal{E}}_n \to 0$ and $\Delta_n \to 1$).

The modewise energy balance Eq. (25) can be applied to quantify the partition between the kinetic and potential energies. Taking the real and imaginary part of Eq. (25) one finds

$$\begin{cases} \bar{T}_n - \mathrm{Re}(\Delta_n)\bar{V}_n = 0, \\ \mathrm{Im}(\Delta_n)\bar{V}_n \pm \bar{\mathcal{E}}_n = 0. \end{cases} \quad (27)$$

Thus, the physical meaning of Δ_n is clear: $\mathrm{Re}(\Delta_n)$ represents the coefficient of energy partition between the kinetic and potential energies, while $\mathrm{Im}(\Delta_n)$ quantifies the proportion between viscous energy losses and the potential energy for the n'th mode, i.e.,

$$\Delta_n = \frac{\bar{T}_n}{\bar{V}_n} \mp i\frac{\bar{\mathcal{E}}_n}{\bar{V}_n}. \quad (28)$$

In some particular cases, explicit expressions for Δ_n that do not involve the eigenvectors can be obtained. When β_{ij} vanishes,

$$\Delta_n = \frac{k_n^2}{|k_n|^2} = \frac{k_n}{k_n^*}, \quad (29)$$

and in the case of single degree of freedom

$$\Delta = \frac{\alpha k^2 + \beta}{\alpha|k|^2 + \beta}. \quad (30)$$

Thus, in certain cases, the information about the energy partition in nonconservative systems can be extracted directly from the dispersion relation. This includes the cases when the values of coefficients Δ_n are provided by either Eqs. (29) and (30). In general, the eigenvectors are necessary to evaluate Δ_n according to definition (26).

Finally, it can be noted that for the dynamical systems under consideration, $Re(\Delta_n)$ is always less than unity, which indicates that in the case of spatially damped waves

$$\bar{T}_n < \bar{V}_n. \quad (31)$$

B. Temporally damped waves

The results for temporally damped waves can be recovered in a similar manner. According to Eq. (A8) the expression for the energy losses in one wavelength is

$$2\bar{D}_n \frac{2\pi}{\omega_{nR}} = \pi \frac{|\omega_n|^2}{\omega_{nR}} b_{ij} q_i^n q_j^{n*}, \tag{32}$$

and again the following quantity, $\bar{\mathcal{E}}_n$, is introduced to denote $1/4\pi$ fraction of viscous energy loss in one wavelength,

$$\bar{\mathcal{E}}_n = \bar{D}_n/\omega_{nR} = \frac{1}{4}\frac{|\omega_n|^2}{\omega_{nR}} b_{ij} q_i^n q_j^{n*}. \tag{33}$$

According to the results for the average over wavelength quantities provided in (A2), (19), and (33) one may find (no summation over repeated n's)

$$-\omega_n^2 \bar{T}_n + |\omega_n|^2 \bar{V}_n \pm i\omega_n \omega_{nR} \bar{\mathcal{E}}_n = 0, \tag{34}$$

what represents the modewise energy balance in the case of temporally damped waves.

The real and imaginary parts of Eq. (34) are, respectively,

$$\begin{aligned} -(\omega_{nR}^2 - \omega_{nI}^2)\bar{T}_n + |\omega_n|^2 \bar{V}_n \mp \omega_{nR}\omega_{nI}\bar{\mathcal{E}}_n = 0, \\ -2\omega_{nI}\bar{T}_n \pm \omega_{nR}\bar{\mathcal{E}}_n = 0, \end{aligned} \tag{35}$$

and equipartition of energy obviously follows from the above relations,

$$\bar{T}_n = \bar{V}_n. \tag{36}$$

IV. COMPLEX GROUP VELOCITY

This section investigates the possibility of extrapolation of the energetic definition (1) of the group velocity to the case of absorbing media. Both cases of spatial and temporal attenuation are examined.

A. Adiabatic approximation

In the following, the approximate expressions for the complex group velocities are sought based on the assumption that the corresponding eigenvectors A_i^n are sufficiently slowly varying functions of frequency (wave number) in the case of spatial (temporal) attenuation. The validity of the assumption is subsequently discussed.

In the case of spatially damped waves analysis stems from the Eq. (17) which can also be recast in the form,

$$[-\omega^2 \rho_{ij} + \alpha_{ij} k_n^2 + \beta_{ij} \pm i\omega b_{ij}] A_i^n(\omega) A_j^{n*}(\omega) = 0, \tag{37}$$

involving normalized eigenvectors $A_i^n(\omega)$.

Differentiating the Eq. (37) with respect to ω and neglecting the terms involving derivatives of eigenvectors one arrives at the following expression (no summation over repeated n's),

$$\left[-2\omega\rho_{ij} + 2\alpha_{ij}k_n \frac{dk_n}{d\omega} \pm i b_{ij}\right] A_i^n(\omega) A_j^{n*}(\omega) \approx 0, \tag{38}$$

which further allows the derivation of the approximate expression for the complex group velocity,

$$C_g^n \approx \frac{[2\alpha_{ij}k_{nR} + 2i\alpha_{ij}k_{nI}] A_i^n(\omega) A_j^{n*}(\omega)}{[2\omega\rho_{ij} \mp i b_{ij}] A_i^n(\omega) A_j^{n*}(\omega)}. \tag{39}$$

The identity obtained from the imaginary part of Eq. (37),

$$\left[2a_{ij}k_{nI} \pm \frac{\omega}{k_{nR}} b_{ij}\right] A_i^n(\omega) A_j^{n*}(\omega) = 0. \tag{40}$$

allows Eq. (39) to be rewritten as

$$C_g^n \approx \frac{[2\alpha_{ij}k_{nR} \mp i\omega b_{ij}/k_{nR}] A_i^n(\omega) A_j^{n*}(\omega)}{[2\omega\rho_{ij} \mp i b_{ij}] A_i^n(\omega) A_j^{n*}(\omega)}. \tag{41}$$

The energetic interpretation of the above expression is now straightforward. To proceed, we invoke the expressions for the average over a period kinetic energy (A1), energy flux (A3), and power dissipation (A4), so that it remains to multiply the numerator and denominator in Eq. (39) by $\omega \exp(-2k_I x)/4$ to derive,

$$C_g^n \approx \frac{\bar{F}_n \pm i c_\phi^n \bar{\mathcal{E}}_n}{2\bar{T}_n \pm i\bar{\mathcal{E}}_n}, \tag{42}$$

where c_ϕ^n denotes the phase velocity and the plus (minus) sign is selected for the waves traveling in positive (negative) x direction.

The analysis for the temporally damped modes is completely analogous and follows from the Eq. (19),

$$[-\omega_n^2 \rho_{ij} + \alpha_{ij} k^2 + \beta_{ij} \pm i\omega_n b_{ij}] A_i^n(k) A_j^{n*}(k) = 0. \tag{43}$$

Under the assumption of slowly varying eigenvectors, differentiation of the Eq. (43) with respect to k provides

$$\left[-2\rho_{ij}\omega_n \frac{d\omega_n}{dk} + 2\alpha_{ij}k \pm i b_{ij} \frac{d\omega_n}{dk}\right] A_i^n(k) A_j^{n*}(k) \approx 0, \tag{44}$$

and consequently, the expression for the group velocity is

$$C_g^n \approx \frac{2\alpha_{ij}k A_i^n(k) A_j^{n*}(k)}{[2\omega_n\rho_{ij} \mp i b_{ij}] A_i^n(k) A_j^{n*}(k)}. \tag{45}$$

With the use of the identity obtained by evaluation of the imaginary part of Eq. (43) this further simplifies to

$$C_g^n \approx \frac{\alpha_{ij}k A_i^n(k) A_j^{n*}(k)}{\omega_{nR}\rho_{ij} A_i^n(k) A_j^{n*}(k)}. \tag{46}$$

The energetic interpretation of Eq. (46) requires the expressions for the averaged over a wavelength kinetic energy (A5) and energy flux (A7). For traveling modes ($\omega_{nR} \neq 0$), the following approximate relation can be obtained,

$$C_g^n \approx \frac{|\omega_n|^2}{\omega_{nR}^2} \frac{\bar{F}_n}{2\bar{T}_n}, \tag{47}$$

or alternatively, taking into account the equipartition of energy for temporally attenuated modes (36),

$$C_g^n \approx \frac{|\omega_n|^2}{\omega_{nR}^2} C_e^n, \tag{48}$$

where C_e^n denotes the energy velocity. Note, that the factor $|\omega_n|^2/\omega_{nR}^2$ in Eq. (48) may be neglected in the case of traveling waves with e folding time greater than a period, however, should be preserved in the case of rapidly damped waves.

First of all, it is important to stress that the eigenvectors in Eqs. (37) and (43), provided $b_{ij} \neq 0$, are parameter independent only in the case of weak coupling, so that the cross-coupling entries in Eqs. (8)–(10) can be eliminated. Moreover, this appears to be the only case when Eqs. (42) and (48) are in fact the exact energetic interpretations of the group velocities, i.e.,

$$C_g^n = \frac{\bar{F}_n \pm i c_\phi^n \bar{\mathcal{E}}_n}{2\bar{T}_n \pm i\bar{\mathcal{E}}_n}, \quad C_g^n = \frac{|\omega_n|^2}{\omega_{nR}^2} C_e^n. \tag{49}$$

Thus, approximations (42) and (48) neglect coupling effect, and therefore, can be regarded as adiabatic approximations, in analogy with those known from quantum mechanics and acoustics. As a consequence, group velocities in the case of single degree of freedom dynamical system (8)–(10) (Klein-Gordon equation with damping, considered further in Sec. V) yield,

$$C_g = \frac{\bar{F} \pm i c_\phi \bar{\mathcal{E}}}{2\bar{T} \pm i\bar{\mathcal{E}}}, \quad C_g = \frac{|\omega|^2}{\omega_R^2} C_e. \tag{50}$$

Second, adiabatic approximations may be accurate even despite of significant coupling, as for instance in the case of longitudinal modes in Biot's theory (considered further in Sec. VII).

Finally, it should be mentioned that the energetic interpretations (49) and (50) recover the energetic definition of the group velocity (1) in the limiting case of no dissipation.

B. Dissipation outflux

The adiabatic approximation in the case of spatially attenuated waves (42) can alternatively be written as

$$C_g \approx \frac{\bar{F} \pm i\bar{F}_{\mathcal{E}}}{2\bar{T} \pm i\bar{\mathcal{E}}}, \tag{51}$$

where $\bar{F}_{\mathcal{E}} = c_\phi \bar{\mathcal{E}}$. The term $\bar{F}_{\mathcal{E}}$, the energy flux density on dimensional grounds, appears to be the product of the energy dissipated $\bar{\mathcal{E}}$ and the phase velocity $c_\phi = \omega/k_R$. It is important to stress that $\bar{F}_{\mathcal{E}}$ does not represent the energy flux in its usual sense as long as the energy generated by the viscous friction is immediately extracted from the system and does not propagate in a wavelike manner through the medium. This is generally true by virtue of idealization (e.g., in viscous fluid dynamics heat waves are neglected). Thus, in analogy with the mechanical energy flux $\bar{F}$, $\bar{F}_{\mathcal{E}}$ can be regarded as the energy dissipation outflux characterized by some energy dissipation velocity c_ϕ.

An analogous term is absent in the case of temporal damping (42). A more detailed discussion of dissipation outflux is provided in Sec. VIII.

V. KLEIN-GORDON EQUATION WITH DAMPING

To illustrate the preceding considerations we consider the example of the damped KGD with constant coefficients [5],

$$\phi_{tt} - \alpha^2 \nabla^2 \phi + \beta^2 \phi + b\phi_t = 0, \tag{52}$$

where $\phi = \phi(\vec{x}, t)$, and extend the energetic interpretations (50) to more than one dimension.

The energy conservation form (5) for the KGD equation reads,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\phi_t^2 + \frac{1}{2}\alpha^2 \phi_{x_i}^2 + \frac{1}{2}\beta^2 \phi^2\right) + \frac{\partial}{\partial x_i}(-\alpha^2 \phi_t \phi_{x_i}) + b\phi_t^2 = 0, \tag{53}$$

while the dispersion relation and the expression for the complex group velocity are, respectively,

$$-\omega^2 + \alpha^2 \vec{k}^2 + \beta^2 - ib\omega = 0, \tag{54}$$

$$\vec{C}_g = \frac{2\alpha^2 \vec{k}}{2\omega + ib}. \tag{55}$$

We first restrict our attention to the case of rightward propagating, spatially damped waves ($\omega_I = 0$, $\vec{k}_R \neq 0$, $\vec{k}_I \neq 0$) and consider a traveling damped wave solution of the form

$$\phi \sim A e^{-\vec{k}_I \cdot \vec{x}} \cos(\vec{k}_R \cdot \vec{x} - \omega t + \eta) = A e^{-\psi(\vec{x})} \cos[\theta(\vec{x}, t) + \eta], \tag{56}$$

where θ denotes the phase, ψ corresponds to spatial attenuation and η is the phase shift,

$$\vec{k}_R = \frac{\partial \theta}{\partial \vec{x}}, \quad \vec{k}_I = \frac{\partial \psi}{\partial \vec{x}}, \quad \omega = -\frac{\partial \theta}{\partial t}. \tag{57}$$

The average over a period kinetic energy is given by

$$\bar{T} = \frac{1}{2}\overline{A^2 \omega^2 e^{-2\psi} \sin^2(\theta + \eta)} = \frac{1}{4} A^2 \omega^2 e^{-2\psi}, \tag{58}$$

while the average power dissipation can be found as

$$2\bar{D} = \overline{A^2 b \omega^2 e^{-2\psi} \sin^2(\theta + \eta)} = \frac{1}{2} A^2 b \omega^2 e^{-2\psi}, \tag{59}$$

so that the expression for the average energy loss is

$$\bar{\mathcal{E}} = \bar{D}/\omega = \frac{1}{4} A^2 b \omega e^{-2\psi}. \tag{60}$$

According to Eq. (53) the expression for the average energy flux is now a vector quantity,

$$\vec{F} = -\overline{A^2 \alpha^2 e^{-2\psi} \omega \sin(\theta + \eta)}\,\overline{[\vec{k}_I \cos(\theta + \eta) - \vec{k}_R \sin(\theta + \eta)]}$$
$$= \frac{1}{2} A^2 \alpha^2 \omega \vec{k}_R e^{-2\psi}. \tag{61}$$

To establish the connection between the complex group velocity and energy transport characteristics we rearrange (55) as follows:

$$\vec{C}_g = \frac{2\alpha^2\vec{k}_R + 2i\alpha^2\vec{k}_I}{2\omega + ib} = \frac{2\alpha^2\vec{k}_R\omega + 2i\alpha^2\vec{k}_I\omega}{2\omega^2 + ib\omega}, \tag{62}$$

and consider the imaginary part of the KGD dispersion relation (54),

$$2\alpha^2\vec{k}_R \cdot \vec{k}_I - b\omega = 0, \tag{63}$$

or alternatively,

$$2\alpha^2\vec{k}_R \cdot \vec{k}_I - b\omega\frac{\vec{k}_R \cdot \vec{k}_R}{|\vec{k}_R|^2} = 0, \tag{64}$$

so that,

$$2\alpha^2\vec{k}_I = b\frac{\omega\vec{k}_R}{|\vec{k}_R|^2} = b\vec{c}_\phi. \tag{65}$$

Finally, we rewrite Eq. (62) in the form,

$$\vec{C}_g = \frac{2\alpha^2\vec{k}_R\omega + i\vec{c}_\phi b\omega}{2\omega^2 + ib\omega}, \tag{66}$$

and multiply the numerator and denominator by $A^2e^{-2\psi}/4$ to obtain

$$\vec{C}_g = \frac{\vec{F} + i\vec{c}_\phi\bar{\mathcal{E}}}{2\bar{T} + i\bar{\mathcal{E}}} = \frac{\vec{F} + i\vec{F}_\mathcal{E}}{2\bar{T} + i\bar{\mathcal{E}}}, \tag{67}$$

which demonstrates the validity of the energetic interpretation (50) in more than one dimension.

Now consider the energetic interpretation of the group velocity in the case of temporally damped waves ($\omega_I \neq 0$, $\vec{k}_R \neq 0$, $\vec{k}_I = 0$). In this case the traveling damped wave solution is of the form (with $\omega_I < 0$ in the case of absorbing media)

$$\phi \sim Ae^{\omega_I t}\cos(\vec{k}\vec{x} - \omega_R t + \eta), \tag{68}$$

and one may recover the following expressions for the averaged over a wavelength kinetic energy and energy flux,

$$\bar{T} = \frac{1}{4}A^2|\omega|^2e^{2\omega_I t}, \tag{69}$$

$$\vec{F} = \frac{1}{2}\alpha^2A^2\omega_R\vec{k}e^{2\omega_I t}. \tag{70}$$

The expression for the group velocity (54) now simplifies to [with the use of the identity obtained by evaluation of the imaginary part of the dispersion relation (54)]

$$\vec{C}_g = \frac{2\alpha^2\vec{k}}{2\omega_R + i(2\omega_I + b)} = \frac{\alpha^2\vec{k}}{\omega_R}. \tag{71}$$

It now suffices to compare Eqs. (69)–(71) to establish an exact representation,

$$\vec{C}_g = \frac{|\omega|^2}{\omega_R^2}\frac{\vec{F}}{2\bar{T}}, \tag{72}$$

and furthermore, taking into account the equipartition of energy, $\bar{T} = \bar{V}$ [which can be easily verified in the three-dimensional case with the use of the identity $|\omega|^2 = \alpha^2\vec{k}^2 + \beta^2$, derived from the dispersion relation (54)],

$$\vec{C}_g = \frac{|\omega|^2}{\omega_R^2}\vec{C}_e. \tag{73}$$

Again, excluding rapidly damped waves in Eq. (73), the group velocity accurately approximates the energy velocity, $\vec{C}_g \approx \vec{C}_e$.

The result for temporally attenuated waves can be related to the behavior of a Gaussian wave packet. According to Eqs. (3) and (4), the one-dimensional version of Eq. (73) suggests that the central wave number k_c is conserved, even though the position of the spatial maximum $x_M = C_g(k_c)t$ is always ahead of the location predicted by the energy velocity, $C_e t$.

VI. ELECTROMAGNETIC WAVES IN A PARTIALLY CONDUCTING MEDIUM. MAXWELL'S EQUATIONS

The physical interpretations of the group velocity in absorbing media developed in Sec. IV have established for a special class of mechanical systems (8)–(10). Maxwell's equations for a partially conducting medium do not fit into this formalism. Nevertheless, completely analogous relations can be obtained.

Consider Maxwell's equations for a homogeneous, isotropic, linear, partially conducting medium in one dimension [13],

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2} + \mu\sigma\frac{\partial E_x}{\partial t}, \tag{74}$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu\epsilon\frac{\partial^2 H_y}{\partial t^2} + \mu\sigma\frac{\partial H_y}{\partial t}, \tag{75}$$

where μ, ϵ, and σ denote permeability, permittivity, and conductivity, respectively.

Assuming solutions in the form of traveling, spatially damped waves,

$$E_x = E_0e^{i(kz-\omega t)}, \quad H_y = H_0e^{i(kz-\omega t)}, \tag{76}$$

where $k = k_R + ik_I$, we note that from the equation

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \tag{77}$$

it follows that $ikE_0 = i\omega\mu H_0$.

The dispersion relation can be written down as

$$-k^2 + \mu\epsilon\omega^2 + i\mu\sigma\omega = 0, \tag{78}$$

or, in alternative form,

$$-k_R^2 + k_I^2 + \mu\epsilon\omega^2 = 0,$$
$$2k_R k_I - \mu\sigma\omega = 0, \tag{79}$$

and provides the following expression for the complex group velocity,

$$C_g = \frac{2k}{2\mu\epsilon\omega + i\mu\sigma}. \tag{80}$$

The expressions for the average electric and magnetic energy densities are, respectively,

$$\bar{W}_e = \frac{1}{2}\epsilon\mathbf{E}\mathbf{E}^* = \frac{1}{2}\epsilon E_x E_x^* = \frac{1}{2}\epsilon|E_0|^2 e^{-2k_I z}, \tag{81}$$

$$\bar{W}_m = \frac{1}{2}\mu\mathbf{H}\mathbf{H}^* = \frac{1}{2}\mu H_y H_y^* = \frac{1}{2}\frac{|k|^2}{\omega^2\mu}|E_0|^2 e^{-2k_I z}. \tag{82}$$

The average power flow per unit area equals the real part of the complex Poynting vector,

$$\bar{\mathbf{S}} = \mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \mathrm{Re}(E_x H_y^*)\hat{\mathbf{z}}, \tag{83}$$

$$\bar{S}_z = \frac{k_R}{\omega\mu}|E_0|^2 e^{-2k_I z}. \tag{84}$$

The expression for the average dissipation power density can be found as

$$\bar{P}_d = \frac{1}{2}\mathbf{E}\cdot\mathbf{J}_c^* = \frac{1}{2}\sigma E_x E_x^* = \frac{1}{2}\sigma|E_0|^2 e^{-2k_I z}, \tag{85}$$

and again we define $1/4\pi$ fraction of the energy loss in one cycle $\bar{\mathcal{E}}$, so that

$$\bar{\mathcal{E}} = \bar{P}_d/\omega = \frac{\sigma}{2\omega}|E_0|^2 e^{-2k_I z}. \tag{86}$$

Using the second Eq. (79) we can rearrange (80) as follows,

$$C_g = \frac{k_R/\mu\omega + ic_\phi\sigma/2\omega}{\epsilon + i\sigma/2\omega}, \tag{87}$$

where $c_\phi = \omega/k_R$ is the phase velocity, so that finally,

$$C_g = \frac{\bar{S}_z + ic_\phi\bar{\mathcal{E}}}{2\bar{W}_e + i\bar{\mathcal{E}}} = \frac{\bar{S}_z + i\bar{F}_\mathcal{E}}{2\bar{W}_e + i\bar{\mathcal{E}}}, \tag{88}$$

where the quantity $\bar{F}_\mathcal{E} = c_\phi\bar{\mathcal{E}}$, in analogy with previous considerations, is recognized as the Joule dissipation outflux.

It can be easily verified that the energy partition coefficient Δ is given by

$$\Delta = \frac{k^2}{|k|^2} = \frac{\bar{W}_e}{\bar{W}_m} + i\frac{\bar{\mathcal{E}}}{\bar{W}_m}, \tag{89}$$

so that $\mathrm{Re}(\Delta)$ provides the ratio between the electric and magnetic energy densities, while $\mathrm{Im}(\Delta)$ quantifies the proportion between the energy loss in one period and magnetic density. In conducting media the magnetic energy density is known to be always larger than the electric energy density in the case of spatial attenuation [13] which also follows from Eq. (89).

One may also obtain the identity for the temporally damped waves. In this case the expression for the group velocity is

$$C_g = \frac{k}{\mu\epsilon\omega_R}. \tag{90}$$

The expressions for the energy densities and power flow averaged over a wavelength are

$$\bar{W}_e = \frac{1}{2}\epsilon|E_0|^2 e^{2\omega_I z}, \tag{91}$$

$$\bar{W}_m = \frac{1}{2}\frac{k^2}{|\omega|^2\mu}|E_0|^2 e^{2\omega_I z}, \tag{92}$$

$$\bar{S}_z = \frac{k\omega_R}{\mu|\omega|^2}|E_0|^2 e^{2\omega_I z}. \tag{93}$$

The equipartition of electric and magnetic energy densities, $\bar{W}_e = \bar{W}_m$, can be confirmed with the use of dispersion relation (78).

Consequently, one arrives at

$$C_g = \frac{|\omega|^2}{\omega_R^2}\frac{\bar{S}_z}{2\bar{W}_e} = \frac{|\omega|^2}{\omega_R^2}C_e. \tag{94}$$

VII. LONGITUDINAL WAVES IN A MULTIPHASE CONTINUUM. BIOT'S THEORY

The approach to the complex group velocity developed above applies to a multiphase continuum, in particular, to Biot's theory of wave propagation in an isotropic, homogeneous, porous solid [14]. Biot's theory implies purely viscous solid-fluid interphase interactions in the low frequency range and viscoelastic interphase interactions in the higher frequency range. The governing equations in the low frequency range can be shown to satisfy the formalism (8)–(10) and approximate physical interpretations of the complex group velocities (42), Eq. (48) apply directly to both longitudinal modes. In the general, full frequency range theory it is necessary to include viscoelastic effects which introduce certain modifications to the earlier obtained result (42). Analysis is provided for the case of spatially damped waves propagating in the negative x direction to maintain consistency with Biot's original work [14].

According to low frequency Biot's theory

$$2T = \rho_{11}\dot{u}_i^2 + 2\rho_{12}\dot{u}_i\dot{U}_i + \rho_{22}\dot{U}_i^2, \tag{95}$$

$$2V = \sigma_{ij}e_{ij} + s\varepsilon, \tag{96}$$

$$2D = b(\dot{u}_i - \dot{U}_i)^2, \tag{97}$$

where u and U are the displacements of the solid and fluid phases, respectively, ρ_{ij} is the mass matrix whose diagonal (off-diagonal) components represent reference phase densities (added mass effects); $b = \phi^2\eta_f/K$, where ϕ, η_f, and K

denote porosity, viscosity and permeability, respectively.

The solid and fluid strains are defined as follows:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon = U_{i,i}, \tag{98}$$

and the constitutive equations thus read

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + Q\varepsilon\,\delta_{ij}, \tag{99}$$

$$s = Qe_{kk} + R\varepsilon. \tag{100}$$

Further analysis is provided for the one-dimensional case and thus restricted to longitudinal waves. In this case we find the following expressions for the potential energy (96) in terms of generalized coordinates,

$$2V = (\lambda + 2\mu)u_x^2 + 2Qu_xU_x + RU_x^2, \tag{101}$$

which, along with Eqs. (95) and (97), represent a particular case of the formalism (8)–(10). The poroacoustic Poynting vector

$$P_i = -\sigma_{ij}\dot{u}_j - s\dot{U}_i, \tag{102}$$

reduces to a scalar expression for the energy flux [consistent with the general expression (14)],

$$F = -\sigma\dot{u} - s\dot{U} = -(\lambda + 2\mu)u_x\dot{u} - Q(u_x\dot{U} + U_x\dot{u}) - RU_x\dot{U}. \tag{103}$$

To illustrate the steps in the approach for coupled systems we consider the quadratic forms

$$2T' = \rho_{11}u^2 + 2\rho_{12}uU + \rho_{22}U^2, \tag{104}$$

$$2V' = k^2[(\lambda + 2\mu)u^2 + 2QuU + RU^2], \tag{105}$$

$$2D' = b(u^2 - 2uU + U^2), \tag{106}$$

and derive the governing equations in frequency wavenumber space according to Eq. (15),

$$-\omega^2(\rho_{11}u + \rho_{12}U) + k^2[(\lambda + 2\mu)u + QU] + \Xi = 0, \tag{107}$$

$$-\omega^2(\rho_{12}u + \rho_{22}U) + k^2(Qu + RU) - \Xi = 0, \tag{108}$$

$$\Xi = i\omega b\mathcal{F}(u - U). \tag{109}$$

An *ad hoc* frequency dependent correction factor $\mathcal{F}$ [14] is introduced here in Eq. (109). Introduction of the complex viscosity $\mathcal{F}\eta_f$ includes viscoelastic effects and physically describes the lag between the filtration velocity and the shear stress exerted on the pore wall [14]. While the real part of the complex viscosity corresponds to the viscous interaction, the imaginary part takes into account the purely elastic response. By taking the viscoelastic interaction into account one can investigate a more general acoustic porous media model that is valid for both low- and high-frequency regimes. The more general results for the full frequency range problem must recover the results derived above for the low-frequency theory in the limiting case $\mathcal{F} \to 1$.

The eigenvalue problem (107), (108) for k^2, can be classified as a standard eigenvalue problem (or degenerate quadratic eigenvalue problem for k) and admits a simple analytic solution [14] (unlike the corresponding problem for ω, which is strongly nonlinear; this case of temporal attenuation is beyond the scope of the present analysis). Two possible longitudinal modes exist in Biot's theory, the so-called P1 and P2 waves, which we take to correspond to the eigenvalues k_1, k_2, and eigenvectors (u_1, U_1) and (u_2, U_2), respectively.

In this notation, the modewise, averaged over a period values of the kinetic energy, potential energy, energy flux, and D_n are $(n=1,2)$,

$$\bar{T}_n = \frac{1}{4}\omega^2[\rho_{11}u_nu_n^* + \rho_{12}(u_n^*U_n + u_nU_n^*) + \rho_{22}U_nU_n^*], \tag{110}$$

$$\bar{V}_n = \frac{1}{4}|k_n|^2[(\lambda + 2\mu)u_nu_n^* + Q(u_n^*U_n + u_nU_n^*) + RU_nU_n^*], \tag{111}$$

$$\bar{F}_n = \frac{1}{2}\omega\,\mathrm{Re}(k_n)[(\lambda + 2\mu)u_nu_n^* + Q(u_n^*U_n + u_nU_n^*) + RU_nU_n^*], \tag{112}$$

$$2\bar{D}_n = \frac{1}{2}\omega^2 b\mathcal{F}_R(u_nu_n^* - u_n^*U_n - u_nU_n^* + U_nU_n^*). \tag{113}$$

Viscoelastic interaction implies elastic energy storage during each cycle as well as viscous energy losses. Upon introduction of the frequency correction factor in Eq. (113) the average quantity $2\bar{D}_n$ becomes complex. It is now the real part of $2\bar{D}_n$ that corresponds to the viscous power dissipation, while the imaginary part of $2\bar{D}_n$ serves to quantify the elastic energy stored. Indeed, the energy stored can be expressed as the product of the elastic part of the force exerted by the fluid on the solid in the x direction, times the relative solid displacement

$$\text{Energy stored} = -b\mathcal{F}_I(\dot{U}_R - \dot{u}_R)(u_R - U_R). \tag{114}$$

Thus its maximum value over one cycle is (here the subscript root-mean square stands for root-mean-square values),

$$\text{Max energy stored} = b\mathcal{F}_I(\dot{U}_R - \dot{u}_R)_{\mathrm{rms}}(U_R - u_R)_{\mathrm{rms}}, \tag{115}$$

which can be reinterpreted in terms of contributions due to P1 and P2 waves in the form $(n=1,2)$,

$$\frac{1}{2}\sum_n \omega b\mathcal{F}_I(u_nu_n^* - u_n^*U_n - u_nU_n^* + U_nU_n^*). \tag{116}$$

A. Complex group velocity

The approximate energetic interpretations of the complex group velocities for the longitudinal waves are sought in the

following and these now include an additional term due to elastic energy stored.

As was mentioned above, Biot's theory admits two possible longitudinal modes. Each characteristic solution ($n=1,2$) must satisfy (17), which is now obtained by multiplication of Eqs. (107) and (108) by u^* and U^*, respectively, and adding the two equations,

$$-\omega^2[\rho_{11}u_nu_n^* + \rho_{12}(u_n^*U_n + u_nU_n^*) + \rho_{22}U_nU_n^*] + k_n^2[(\lambda+2\mu)u_nu_n^* + Q(u_n^*U_n + u_nU_n^*) + RU_nU_n^*] + i\omega b\mathcal{F}(u_nu_n^* - u_n^*U_n - u_nU_n^* + U_nU_n^*) = 0. \quad (117)$$

In the form of the modewise energy balance Eq. (25) this reads

$$-\bar{T}_n + \Delta_n\bar{V}_n + i\bar{\mathcal{E}}_n' - \bar{\mathcal{E}}_n'' = 0, \quad \Delta_n = k_n^2/|k_n|^2 \quad (118)$$

where $\bar{\mathcal{E}}_n'$ and $\bar{\mathcal{E}}_n''$,

$$\begin{pmatrix}\bar{\mathcal{E}}_n' \\ \bar{\mathcal{E}}_n''\end{pmatrix} = \frac{1}{4}\omega b\begin{pmatrix}\mathcal{F}_R \\ \mathcal{F}_I\end{pmatrix}(u_nu_n^* - u_n^*U_n - u_nU_n^* + U_nU_n^*), \quad (119)$$

are, respectively, $1/4\pi$ fraction of the energy loss due to the viscous dissipation and half the maximum elastic energy stored in one cycle modewise. In particular, according to Eqs. (118) and (119), the energy partition between the kinetic and potential energy, and the energy losses and potential energy can be quantified as follows,

$$\frac{\bar{T}_n}{\bar{V}_n} = \mathrm{Re}(\Delta_n) + \frac{\mathcal{F}_I}{\mathcal{F}_R}\mathrm{Im}(\Delta_n), \quad \frac{\bar{\mathcal{E}}_n'}{\bar{V}_n} = -\mathrm{Im}(\Delta_n). \quad (120)$$

With the above remarks the derivation for the group velocity interpretation can be recovered in the same manner as in Sec. IV. In this case, for leftward propagating waves, we obtain the following adiabatic approximations

$$C_g^n \approx \frac{\bar{F}_n - ic_\phi^n\bar{\mathcal{E}}_n'}{2\bar{T}_n + \Psi''\bar{\mathcal{E}}_n'' - i\Psi'\bar{\mathcal{E}}_n'},$$

$$\Psi' = 1 + \frac{\omega}{\mathcal{F}_R}\frac{d\mathcal{F}_R}{d\omega}, \quad \Psi'' = 1 + \frac{\omega}{\mathcal{F}_I}\frac{d\mathcal{F}_I}{d\omega}. \quad (121)$$

The physical interpretation of the above statement is clear. The term $\bar{F}_{\mathcal{E}n} = c_\phi^n\bar{\mathcal{E}}_n'$ represents the dissipation outflux due to the viscous interphase interactions as before, while $\bar{\mathcal{E}}_n'$ and $\bar{\mathcal{E}}_n''$ denote viscous energy losses and half of the maximum of the elastic energy stored, respectively. Ψ'' and Ψ' are nondimensional coefficients expressed in terms of the frequency correction factor. The results for the low frequency Biot's theory (42) can be recovered from Eq. (121) in the limiting case $\mathcal{F}\to 1$.

Moreover, in this particular case the adiabatic approximation (121) is also a leading order approximation for both the real and imaginary parts of the complex group velocity. In the high frequency range the small parameter in the eigenvalue problem (107), (108) is $\mathcal{F}/\omega$. As long as $\mathrm{Re}\,\mathcal{F} = O(\omega^{1/2})$ and $\mathrm{Im}\,\mathcal{F} = O(\omega^{1/2})$ [14], the perturbation is of order $O(\omega^{-1/2}) + iO(\omega^{-1/2})$. Thus, k_n and C_g^n can be expanded in powers $\omega^{1/2}$,

$$k_n/\omega = \lambda_{n0} + i\lambda_{n1}/\omega^{1/2} + \lambda_{n2}/\omega^{1/2} + i\lambda_{n3}/\omega + \dots,$$

$$C_g^n = C_g^{n0} + C_g^{n1}/\omega^{1/2} + iC_g^{n2}/\omega^{1/2} + C_g^{n3}/\omega + \dots \quad (122)$$

Further considerations suggest that

$$C_g^n = \frac{\bar{F}_n - ic_\phi^n\bar{\mathcal{E}}_n'}{2\bar{T}_n + \Psi''\bar{\mathcal{E}}_n'' - i\Psi'\bar{\mathcal{E}}_n'} + O\left(\frac{1}{\omega}\right) + iO\left(\frac{1}{\omega}\right), \quad (123)$$

where the first summand contains leading $O(1)$, $O(\omega^{-1/2})$. and $iO(\omega^{-1/2})$ terms.

The validity of the adiabatic approximation (123) is discussed further in Sec. VII B and illustrated with sample parameters for water saturated Berea sandstone.

B. Numerical results

Numerical results are obtained for water saturated Berea sandstone with the physical properties provided in Table I. Generalized poroelastic parameters λ, Q, and R are related to the porosity, the solid and fluid bulk moduli, the bulk modulus of the porous drained matrix and the shear modulus via Gedanken experiments [15]; reference phase densities ρ_{ij} are related to porosity, tortuosity, grain and saturating fluid densities [14] (see Appendix B).

The dispersion relation for the longitudinal modes has the form [14]

$$(q_{11}q_{22} - q_{12}^2)z^2 - (q_{22}\gamma_{11} + q_{11}\gamma_{22} - 2q_{12}\gamma_{12})z + (\gamma_{11}\gamma_{22} - \gamma_{12}^2) + \frac{ib}{\rho\omega}\mathcal{F}(\kappa)(z-1) = 0, \quad (124)$$

where q_{ij} and γ_{ij} are normalized Biot's parameters and normalized reference densities, respectively, $\rho=\rho_{11}+2\rho_{12}+\rho_{22}$, $z=(ck_n/\omega)^2$, $n=1,2$ (c is the characteristic velocity), and $\kappa = \delta(f/f_c)^{1/2}$ with characteristic frequency f_c defined as

$$f_c = \frac{b}{2\pi\rho(\gamma_{12}+\gamma_{22})}. \quad (125)$$

The expression for the frequency correction factor in terms of Bessel-Kelvin zero order functions is [14]

$$\mathcal{F}(\kappa) = \frac{1}{4}\left(\frac{\kappa T(\kappa)}{1 - \frac{2}{i\kappa}T(\kappa)}\right), \quad T(\kappa) = \frac{\mathrm{ber}'(\kappa) + i\mathrm{bei}'(\kappa)}{\mathrm{ber}(\kappa) + i\mathrm{bei}(\kappa)}.$$

Figures 1 and 2 contain the results for the complex group velocities (real and imaginary parts, respectively) calculated both exactly, with the direct use of the dispersion relation (124), and approximately, according to the adiabatic (also the leading order) approximation (123). The results for the energy velocity are also included in Fig. 1. As was mentioned above, the validity of the group velocity approximation strongly depends on certain properties of the eigenvectors. The solution of the eigenvalue problem (107), (108) indicates that eigenvectors indeed exhibit the desired property, not only in the high-frequency range. Results for real and

TABLE I. Physical properties of the porous material (Berea sandstone) and saturating fluid (water).

Porosity			ϕ		0.20	
Permeability (mD)			K		360.00	
Tortuosity			a		2.40	
Structural factor[a]			δ		2.83	
Frame bulk modulus (GPa)			K_b		10.37	
Shear modulus (GPa)			μ		7.02	
Grain bulk modulus (GPa)			K_s		36.50	
Liquid bulk modulus (GPa)			K_f		2.25	
Solid density (kg/m^3)			ρ_s		2644.00	
Liquid density (kg/m^3)			ρ_f		1000.00	
Liquid viscosity (mPa·s)			η_f		1.00	
P1-wave phase velocity[b] (m/s)			V_{p1}^{∞}		3268.20	
P2-wave phase velocity[b] (m/s)			V_{p2}^{∞}		793.33	
Characteristic frequency (kHz)			f_c		6.07	
Characteristic velocity (m/s)			c		3260.26	
Biot parameters (GPa)				Phase densities (kg/m^3)		
λ	μ	Q	R	ρ_{11}	ρ_{12}	ρ_{22}
8.224	7.020	0.982	0.380	2415.2	−300	500

[a]Corresponds to the pore structure [14].
[b]High frequency limit.

imaginary parts of the complex group velocities illustrate the relative error introduced by the truncation of the coupling terms. The approximation is much more accurate in the case of the P2 wave, so that the exact and approximate results are indistinguishable. This indicates that interphase coupling does not significantly effects the propagation of the highly damped P2 wave. In the case of the P1 wave the effect of interphase coupling is more pronounced, especially in the low-frequency range. The approximation is more accurate in the higher-frequency range, as was predicted theoretically. The energy velocity is found to be slower (faster) than the group velocity in the low-(high-)frequency range, however, never exceeds the values of the wavefront velocities, $C_e^1 < V_{p1}^{\infty}$, $C_e^2 < V_{p2}^{\infty}$, in accord with the causality principle.

Figure 3 contains the results for the modewise kinetic energy and energy losses in one period, due to viscous dissipation, $\bar{T}_n$ and $\bar{\mathcal{E}}_n'$, measured in percentage of the potential energies $\bar{V}_n$ in accordance with Eq. (120). For the P1 wave, equipartition of energy, $\bar{T}_1 \approx \bar{V}_1$, holds nearly exactly (with accuracy greater than 0.05%) at all frequencies. For the P2 wave similar results, $\bar{T}_2 \approx \bar{V}_2$, are only valid in the high-frequency range. Significant energy losses are observed in the low-frequency range, especially for the P2 wave, as expected. These results reconfirm, from an energetic point of view, the role of the P2 wave as the main dissipation mechanism in porous solids. Moreover, this serves a vivid example of the application of the modewise energy balance equation, since the results for the energy partition were extracted directly from the dispersion relation (124).

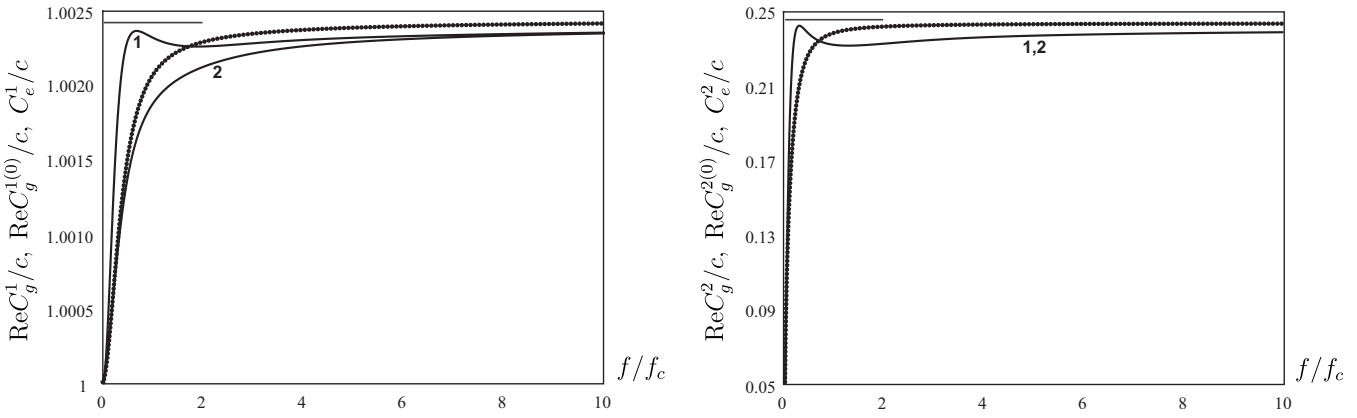

FIG. 1. Group velocities and corresponding adiabatic approximations (real parts) versus frequency (curves labeled 1 and 2, respectively). Energy velocities versus frequency (dotted line). Horizontal lines indicate limiting wavefront velocities V_{p1}^{∞} and V_{p2}^{∞} (Table I). P1-wave results (left), P2-wave results (right).

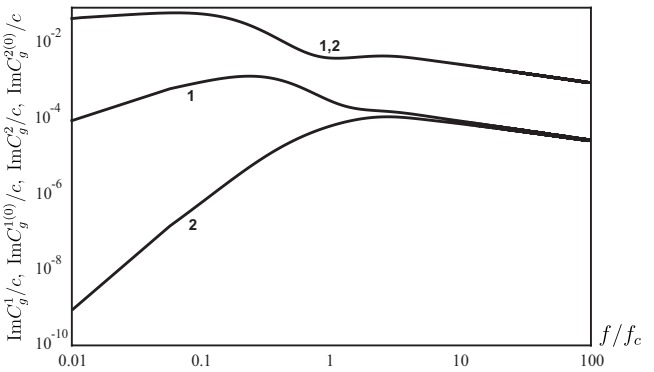

FIG. 2. Group velocities and corresponding adiabatic approximations (imaginary parts) versus frequency (curves labeled 1 and 2, respectively). P1-wave results (lower curves) and P2-wave results (upper curves).

VIII. CONCLUSIONS

In the case of conservative dynamical systems group velocity can be defined both kinematically and from energy principles. In the case of nonconservative systems, a general energetic definition, similar to that for conservative systems, does not exist. Nevertheless, in some exceptional cases, identities relating the complex group velocity and energy transport characteristics can be established for both the case of spatial and the case of temporal attenuation. The exact energetic interpretations have been presented for the KGD Eqs. (67) and (73) and Maxwell's Eqs. (88) and (94). In the latter case the complete analogy with mechanical systems is remarkable, since Maxwell's equations do not satisfy the formalism (8)–(10).

In the case of Biot's theory the results can be summarized as follows. The energetic interpretation in the form of the

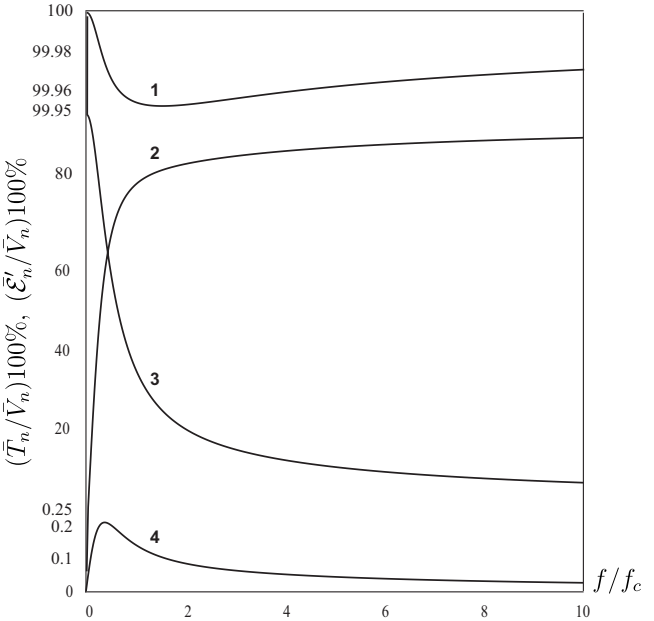

FIG. 3. Kinetic energies, (1) $\bar{T}_1$ and (2) $\bar{T}_2$, and energy losses, (3) $\bar{\mathcal{E}}'_1$ and (4) $\bar{\mathcal{E}}'_2$, measured in percentage of the potential energies $\bar{V}_1$ and $\bar{V}_2$ respectively.

adiabatic approximation established for the complex group velocity (123) is a leading order approximation for both real and imaginary parts of the group velocity. Numerical results suggest a minuscule coupling effect on the propagation of the highly damped slow P2 mode.

All energetic interpretations of the complex group velocity in the case of spatial attenuation contain the dissipation outflux term representing the product of the phase velocity and energy dissipated. This quantity although an energy flux on dimensional grounds, is not the energy flux in its usual sense as long as the energy generated by the internal friction is instantaneously extracted from the medium and does not propagate after it is generated. In all the cases considered, including low- and higher-frequency Biot's theory, the dissipation energy velocity is found to equal the phase velocity, as can be expected for cases in which dissipation is caused by friction. We believe, this can be understood in terms of the following thought experiment. Imagine a chain of masses in a frictionless medium. The first mass begins to move with a constant velocity, hits the second mass, and transfers energy to this mass. The second mass, in turn, moves with a fixed velocity and hits the third mass, setting it in motion, and so on. Assume that the energy is only dissipated during inelastic collisions, which occur instantaneously, and each collision removes only a small fraction of the initial energy. In this simple example of a spatially damped longitudinal wave, heat waves are generated and extracted from the medium during collisions only, and hence, energy is dissipated with the velocity of the individual masses, or the phase velocity. An analogous thought experiment in the case of temporal attenuation assumes intrinsic friction of the medium and purely elastic collisions. In the latter case the energy is extracted permanently, immediately after the first mass is set in motion. There is no characteristic velocity to be associated with the energy dissipated, therefore, the quantity $c_\phi \bar{\mathcal{E}}$ is physically meaningless, and the corresponding term is absent in the energetic interpretation of the group velocity.

Finally, it is important to stress that the analysis and the main conclusions presented herein apply to dynamical systems of a special type, namely those governed by second-order hyperbolic partial differential equations with particular dissipation models (viscous and viscoelastic). Although this covers a variety of realistic physical systems, the question of whether or not the interpretation of the complex group velocity from energy principles can be established in general, or at least for a wider class of problems, remains open.

APPENDIX A: AVERAGED QUANTITIES

1. Spatially damped waves. Averaged quantities

$$\bar{T}_n = \frac{1}{2}\rho_{ij}\overline{\dot{q}^n_{iR}\dot{q}^n_{jR}} = \frac{1}{4}\omega^2\rho_{ij}q^n_i q^{n*}_j, \tag{A1}$$

$$\bar{V}_n = \frac{1}{2}\alpha_{ij}\overline{q^n_{iR,x}q^n_{jR,x}} + \frac{1}{2}\beta_{ij}\overline{q^n_{iR}q^n_{jR}} = \frac{1}{4}(\alpha_{ij}|k_n|^2 + \beta_{ij})q^n_i q^{n*}_j, \tag{A2}$$

$$\bar{F}_n = -\alpha_{ij}\overline{\dot{q}^n_{iR} q^n_{jR,x}} \cdot \mathbf{n} = \frac{1}{2}\omega \,\mathrm{Re}(k_n)\alpha_{ij} q^n_i q^{n*}_j, \tag{A3}$$

$$2\bar{D}_n = b_{ij}\overline{\dot{q}^n_{iR}\dot{q}^n_{jR}} = \frac{1}{2}\omega^2 b_{ij} q^n_i q^{n*}_j. \tag{A4}$$

2. Temporally damped waves. Averaged quantities

$$\bar{T}_n = \frac{1}{2}\rho_{ij}\overline{\dot{q}^n_{iR}\dot{q}^n_{jR}} = \frac{1}{4}|\omega_n|^2 \rho_{ij} q^n_i q^{n*}_j, \tag{A5}$$

$$\bar{V}_n = \frac{1}{2}\alpha_{ij}\overline{q^n_{iR,x} q^n_{jR,x}} + \frac{1}{2}\beta_{ij}\overline{q^n_{iR} q^n_{jR}} = \frac{1}{4}(\alpha_{ij}k^2 + \beta_{ij}) q^n_i q^{n*}_j$$
$$= \frac{1}{4}a_{ij}(k) q^n_i q^{n*}_j, \tag{A6}$$

$$\bar{F}_n = -\alpha_{ij}\overline{\dot{q}^n_{iR} q^n_{jR,x}} \cdot \mathbf{n} = \frac{1}{2}\mathrm{Re}(\omega_n) k \alpha_{ij} q^n_i q^{n*}_j, \tag{A7}$$

$$2\bar{D}_n = b_{ij}\overline{\dot{q}^n_{iR}\dot{q}^n_{jR}} = \frac{1}{2}|\omega_n|^2 b_{ij} q^n_i q^{n*}_j. \tag{A8}$$

APPENDIX B: POROELASTIC MECHANICAL PARAMETERS

Reference phase densities,

$$\rho_{11} = (1-\phi)\rho_s + \phi\rho_f(a-1),$$

$$\rho_{12} = \phi\rho_f(1-a),$$

$$\rho_{22} = a\phi\rho_f. \tag{B1}$$

Generalized poroelastic mechanical parameters,

$$\lambda = K_b - 2\mu/3 + K_f(1-\phi-K_b/K_s)^2/\phi_{\mathrm{eff}},$$

$$Q = \phi K_f(1-\phi-K_b/K_s)/\phi_{\mathrm{eff}},$$

$$R = \phi^2 K_f/\phi_{\mathrm{eff}}, \quad b = \phi^2\eta_f/k,$$

$$\phi_{\mathrm{eff}} = \phi + K_f(1-\phi-K_b/K_s)/K_s. \tag{B2}$$

Characteristic quantities,

$$\rho = \rho_{11} + \rho_{22} + 2\rho_{12},$$

$$H = \lambda + 2\mu + R + 2Q,$$

$$c = \sqrt{H/\rho}. \tag{B3}$$

Nondimensional parameters,

$$\gamma_{11} = \rho_{11}/\rho, \quad \gamma_{12} = \rho_{12}/\rho, \quad \gamma_{22} = \rho_{22}/\rho,$$

$$q_{11} = (\lambda + 2\mu)/H, \quad q_{12} = Q/H, \quad q_{22} = R/H. \tag{B4}$$

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